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On non-relativistic conformal symmetries and invariant tensor fields

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Abstract. The largest symmetry group of the Schrödinger equation, the so-called Schrödinger group, is analysed in connection with invariant 2-forms, 1-forms and (0,2)-symmetric tensors through the Beckers-Harnad-Perroud-Winternitz global method developed in the relativistic conformal context. Invariant fields and potentials are obtained and discussed in the physical context of Schrödinger electromagnetism through the corresponding infinitesimal method. Specific attention is paid to magnetic monopole dynamical symmetries. These results are obtained in correspondence with the subalgebra classification determined by using the Patera-Winternitz-Zassenhaus algorithm. The accent is put on the maximal subalgebras of the Schrödinger algebra.

1. Introduction

Since the BHPW (Beckers *et al* 1978) contribution on *tensor* fields invariant under subgroups of the conformal group of spacetime, numerous extensions and applications of the method have been published in the relativistic context. In particular, different results have been derived on gauge fields (Harnad and Vinet 1978, Harnad *et al* 1979, Vinet 1981, Doneux *et al* 1982, Antoine and Jacques 1984), solutions to Yang-Mills equations (Yang and Mills 1954), on (Dirac) spinor fields (Beckers *et al* 1980, Légaré 1983, Légaré and Harnad 1984), etc (Beckers and Jaminon 1978, Beckers *et al* 1979, Beckers and Sinzinkayo 1982, Beckers and Hussin 1983a, b, 1984, Sinzinkayo and Demaret 1985), besides tensor fields such as 2-forms (electromagnetic tensors), 1-forms (four potentials) and rank-two symmetric tensors (metric tensors, for example) studied in BHPW.

Interesting structures such as $O(4)$, $O(4)\times O(2)$, $O(3)\times O(2,1)$ seen as subgroups of the conformal group of spacetime did play a prominent role in particle physics (see BHPW) as well as in classical electrodynamics (Englefield 1972) and invariant objects under such structures gave very attractive information for mathematical physicists.

Such studies do use fundamental tools of differential geometry (Lie derivatives, 2- and 1-forms, ..., fibre bundle techniques, ...) (Kobayashi and Nomizu 1963) as well

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as interesting information issued from classification of subalgebras essentially obtained following the PWZ algorithm (Patera *et al* 1975). Let us recall that the PWZ work has given a lot of mathematical results on maximal and non-maximal subalgebras of the conformal algebra as well as of physical results (Beckers *et al* 1977, Boyer *et al* 1976) through applications to wave equations with interaction. In particular, the PWZ algorithm combined with tensor fields invariant under the Poincaré subalgebras has led to the study of minimal electromagnetic coupling schemes (Beckers and Hussin 1983a, b) and to the study of constants of motion (Beckers and Hussin 1984, Hussin and Sinzinkayo 1985).

All these approaches and studies can, in the relativistic case, be seen as specific contributions issued from the analysis of the conformal group of spacetime and of its (maximal up to conjugacy under the Poincaré group) subgroups where the BHPW work plays a central role. The question is: 'Do we know the corresponding information at the non-relativistic level?'. The answer is negative and the main purpose of this paper will be to study and to complete this non-relativistic domain.

Let us first recall that in the non-relativistic context, Hagen (1972) and Niederer (1972) have put forward the largest symmetry group of the Schrödinger equation leading to the 'maximal kinematical invariance group of the free Schrödinger equation' or to the so-called 'Schrödinger group'. Hereafter we will call it the Schrödinger group SCH_3 and its algebra will be denoted by sch_3, the number 3 referring to the three spatial dimensions (see also Roman *et al* 1972, Barut 1973, Niederer 1973, 1974). It corresponds to the conformal group of spacetime in the relativistic case, so that we can also speak about a *conformal* Galilean symmetry group. Such a structure and some of its substructures have already been studied (Burdet and Perrin 1972, 1975, Burdet *et al* 1973). More particularly the case of SCH_2 has been analysed (Burdet *et al* 1978) through the PWZ algorithm in order to get all the subalgebras of sch_2 (and of its central extension $\widetilde{sch}_2$), while the corresponding analysis of sch_3 is missing!

Let us secondly come back to the question asked above and consequently propose to solve the missing case sch_3. This purpose will be developed in the following but by limiting ourselves to the same level as the one developed in the relativistic case in BHPW. Effectively, we want to get the maximal (up to conjugacy under the Galilei algebra) subalgebras of sch_3 and then to apply invariance considerations on different tensors (2-forms, 1-forms and symmetric $(0, 2)$-tensors, hereafter called F, A and S respectively). Using global and infinitesimal methods, these invariant tensors will be determined and the non-trivial results will be related to physical situations. In particular the interesting case of magnetic monopoles in a non-relativistic context will be recovered and connected to recent works (Jackiw 1980, Horvathy 1983, D'Hoker and Vinet 1984, 1985). We also will consider a six-dimensional non-maximal subalgebra leading to non-trivial electromagnetic considerations.

Our paper is organised as follows. In § 2 we just recall some fundamental elements on the Schrödinger group of three-dimensional space and time and its algebra. Section 3 is devoted to the construction of the maximal subalgebras of sch_3 through the use of the PWZ algorithm up to conjugacy under the Galilei algebra. Then we essentially study in § 4 the tensor fields F, A and S invariant under those maximal subalgebras by the use of the global method displayed in BHPW. Section 5 contains specific information on Galilei's *and* Schrödinger's electromagnetisms and more particularly on associated transformation laws of the fields F and A as well as on their invariance conditions issued from the infinitesimal method. The comparison with the results of § 4 is realised within the physical context of Schrödinger's electromagnetism. Finally in § 6 we present some comments, in particular, in connection with physical constants of motion.

In order to avoid some confusion, let us already mention here that all physical notions referring to 'electromagnetism' are necessarily seen in the non-relativistic context and more particularly in the so-called magnetic limit of the Maxwell theory (Le Bellac and Lévy-Leblond 1973).

2. The Schrödinger group and its algebra

It is well known (Hagen 1972, Niederer 1972) that the maximal kinematical invariance group of the free Schrödinger equation—the so-called Schrödinger group SCH_3—is a twelve-parameter Lie group corresponding to Galilean transformations supplemented by dilations and expansions. A 5×5 matrix representation of SCH_3 is given by

$$g=\begin{pmatrix} R & T \\ 0 & L \end{pmatrix} \tag{2.1}$$

where R is a 3×3 matrix of SO(3), $T=(\boldsymbol{v}, \boldsymbol{a})$ is a 3×2 real matrix and L is a 2×2 matrix of $SL(2,\mathbb{R})$, $L=\left(\begin{smallmatrix}\varepsilon & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ ($\varepsilon\delta-\beta\gamma=1$, ε, β, γ, $\delta\in\mathbb{R}$). These matrices do act on elements $u=\{u^1, u^2, u^3, u^4, u^5\}$ of $\mathbb{R}^5$ such as

$$u'=gu. \tag{2.2}$$

The action of the Schrödinger group on Newtonian spacetime ($\mathbb{R}^{3+1}$) events $(\boldsymbol{x}, t)$ is obtained through the correspondence ($u^5\neq 0$)

$$x^i=u^i/u^5 \qquad (i=1,2,3) \qquad t=u^4/u^5. \tag{2.3}$$

Conversely, to each event $(\boldsymbol{x}, t)$ on $\mathbb{R}^{3+1}$, we associate a ray in $\mathbb{R}^5$ by the relation

$$(\boldsymbol{x}, t)\rightarrow(\lambda\boldsymbol{x}, \lambda t, \lambda) \qquad \lambda\neq 0 \tag{2.4}$$

two points u and v in $\mathbb{R}^5$ being equivalent iff $u=\lambda v$ for some $\lambda\neq 0$. The transformation law on $(\boldsymbol{x}, t)$ is now given by

$$\boldsymbol{x}'=R\boldsymbol{x}+\boldsymbol{v}t+\boldsymbol{a}/(\gamma t+\delta) \qquad t'=\varepsilon t+\beta/(\gamma t+\delta) \qquad (t\neq-\delta/\gamma). \tag{2.5}$$

The infinitesimal version of (2.5) is then

$$\begin{aligned} \boldsymbol{x}'&=\boldsymbol{x}+\boldsymbol{\theta}\times\boldsymbol{x}+\boldsymbol{v}t+\tfrac{1}{2}\alpha\boldsymbol{x}+ct\boldsymbol{x}+\boldsymbol{a} \\ t'&=t+\alpha t+ct^2+b \end{aligned} \tag{2.6}$$

where we can see that the effect of dilations (α) is not the same on spatial and time coördinates.

The Lie algebra sch_3 can be written as the semi-direct sum (Burdet *et al* 1978)

$$sch_3=(t_3\oplus t_3^*)\,\square\,(so(3)\oplus sl(2,\mathbb{R})) \tag{2.7}$$

and a basis of generators is given by $\boldsymbol{K}$ and $\boldsymbol{P}$ for the Abelian algebras t_3 and t_3^*, by $\boldsymbol{J}$ for so(3) and by H, D, C for $sl(2,\mathbb{R})$. These generators are respectively associated with pure Galilean transformations $(\boldsymbol{v})$, spatial translations $(\boldsymbol{a})$, spatial rotations $(\boldsymbol{\theta})$, time translations (b), dilations (α) and expansions (or one-dimensional special conformal transformations) (c). The commutation relations are given by

$$[P_k, P_l] = [K_k, K_l] = [K_k, P_l] = 0 \tag{2.8}$$

$$[J_k, J_l] = \mathrm{i}\varepsilon_{klm} J_m \tag{2.9}$$

$$[H, D] = \mathrm{i}H \qquad [H, C] = 2\mathrm{i}D \qquad [D, C] = \mathrm{i}C \tag{2.10}$$

$$[H, J_k] = 0 \qquad [D, J_k] = 0 \qquad [C, J_k] = 0 \tag{2.11}$$

$$\begin{aligned} &[J_k, P_l] = \mathrm{i}\varepsilon_{klm} P_m && [J_k, K_l] = \mathrm{i}\varepsilon_{klm} K_m \\ &[H, P_k] = 0 && [H, K_k] = \mathrm{i}P_k \\ &[D, P_k] = -\tfrac{1}{2}\mathrm{i}P_k && [D, K_k] = \tfrac{1}{2}\mathrm{i}K_k \\ &[C, P_k] = -\mathrm{i}K_k && [C, K_k] = 0. \end{aligned} \tag{2.12}$$

A specific realisation of the twelve generators can be explicitly given. We have

$$\begin{aligned} &\boldsymbol{P} = -\mathrm{i}\boldsymbol{\nabla} \qquad \boldsymbol{K} = \mathrm{i}t\boldsymbol{\nabla} \qquad \boldsymbol{J} = -\mathrm{i}\boldsymbol{x} \times \boldsymbol{\nabla} \qquad H = \mathrm{i}\partial_t \\ &D = \mathrm{i}(t\partial_t + \tfrac{1}{2}\boldsymbol{x} \cdot \boldsymbol{\nabla}) \qquad C = \mathrm{i}(t^2\partial_t + t\boldsymbol{x} \cdot \boldsymbol{\nabla}). \end{aligned} \tag{2.13}$$

Let us finally recall that the Schrödinger group admits a non-trivial central extension by $\mathbb{R}$ called $\widetilde{\mathrm{SCH}}_3$ and that the corresponding algebra $\widetilde{\mathrm{sch}}_3$ is obtained by defining new generators $\boldsymbol{K}$ and C realised as

$$\boldsymbol{K} = \mathrm{i}t\boldsymbol{\nabla} + m\boldsymbol{x} \qquad C = \mathrm{i}(t^2\partial_t + t\boldsymbol{x} \cdot \boldsymbol{\nabla}) + (m/2)\boldsymbol{x}^2. \tag{2.14}$$

The commutation relations of $\widetilde{\mathrm{sch}}_3$ are essentially given by (2.8)-(2.12) except the ones between K_i and P_k which are replaced by

$$[K_i, P_k] = \mathrm{i}m\delta_{ik}. \tag{2.15}$$

3. Maximal subalgebras of sch_3

Here let us classify the maximal subalgebras of the Schrödinger algebra sch_3 into conjugacy classes under the Galilei group. In order to do this we have to determine the subalgebras, up to conjugacy, of sch_3 following the PWZ classification method (Patera *et al* 1975) but since we are interested in the maximal ones we effectively have to determine only some of them as will be clear in the following.

Let us recall (Burdet *et al* 1978) that the algebra sch_3 is a semi-direct sum of $n \,\square\, \mathrm{f}$ where $n = \mathrm{t}_3 \oplus \mathrm{t}_3^*$ is an Abelian ideal of dimension 6 and f is identified with $\mathrm{so}(3) \oplus \mathrm{sl}(2, \mathbb{R})$ as is clear from (2.6). Now let us summarise the classification method (Patera *et al* 1975). Firstly, we classify all the subalgebras f_i of f into conjugacy classes under the Galilei group. Secondly, we search for the subspaces n_{ik} of n invariant for each f_i. Thirdly, the splitting subalgebras of sch_3 which are the semi-direct sums $n_{ik} \,\square\, \mathrm{f}_i$ are constructed and finally the non-splitting subalgebras can also be obtained. The only point that we have to solve completely is the first one (§ 3.1). Then we have to consider only some invariant subspaces n_{ik} in order to find all the non-conjugated maximal subalgebras of sch_3, these ones being splitting subalgebras only (§ 3.2).

3.1. *Conjugacy classes of subalgebras of f under the Galilei group*

The only non-trivial subalgebra of so(3) = {J} is evidently so(2) (generated by J_3 for example). Then, let us search for non-equivalent subalgebras of sl(2, ℝ) generated by H, D and C satisfying (2.10). The more general combination of these three generators gives the new one

$$Q = \mu H + \nu D + \tau C \qquad \mu, \nu, \tau \in \mathbb{R}. \tag{3.1}$$

Here, the conjugacy under time translations, contained in the Galilei group, is relevant. Indeed, we have

$$\exp(\mathrm{i}bH)Q\exp(-\mathrm{i}bH) = (\mu - \nu b + \tau b^2)H + (\nu - 2\tau b)D + \tau C. \tag{3.2}$$

Let us discuss this result.

(i) If μ, ν, $\tau \neq 0$, we can choose $\tau = 1$. Then, for $b = \frac{1}{2}\nu$, Q is equivalent to $C + \sigma H$, $\sigma \in \mathbb{R}$. Otherwise, if b is such that $\mu - \nu b + b^2 = 0$ then Q is equivalent to $C + \rho D$, $\rho \in \mathbb{R}$. If μ, $\tau \neq 0$ and $\nu = 0$ (with $\tau = 1$), $C + \sigma H$ becomes equivalent to $C + \rho D$, $\rho \in \mathbb{R}$. If ν, $\tau \neq 0$ and $\mu = 0$ (with $\tau = 1$), $C + \nu D$ becomes equivalent to $C + \sigma^2 H$, $\sigma \in \mathbb{R}$. If μ, $\nu = 0$ and $\tau \neq 0$, we recover the generator C.

(ii) If $\tau = 0$ and $\mu, \nu \neq 0$, we can choose $\nu = 1$ so that, with $b = \mu$, Q becomes equivalent to D. If $\tau = 0$ and $\nu = 0$, we recover H.

Thus, the non-equivalent subalgebras of dimension 1 of sl(2, ℝ) are generated by {H}, {D}, {C} and {$C + \sigma H, \sigma \in \mathbb{R}$}. Using this result we can easily construct the subalgebras of dimension 2: there are only two, i.e. {H, D} and {D, C}. The algebra sl(2, ℝ) admits two non-trivial subalgebras of dimension 2 and three of dimension 1 as well as an infinite family of dimension 1.

By combining this information we obtain all the subalgebras of f listed in table 1.

Table 1. Non-equivalent subalgebras of f ≡ so(3)⊕sl(2, ℝ).

Dimension	Notation and generators
6	$f_1 = \{J, H, D, C\}$
5	$f_2 = \{J, H, D\}$, $f_3 = \{J, D, C\}$
4	$f_4 = \{J, H\}$, $f_5 = \{J, D\}$, $f_6 = \{J, C\}$, $f_7 = \{J, C + \sigma H, \sigma \neq 0\}$, $f_8 = \{J_3, H, D, C\}$
3	$f_9 = \{J\}$, $f_{10} = \{J_3, H, D\}$ $f_{11} = \{J_3, D, C\}$, $f_{12} = \{H, D, C\}$
2	$f_{13} = \{J_3, H\}$, $f_{14} = \{J_3, D\}$, $f_{15} = \{J_3, C\}$, $f_{16} = \{J_3, C + \sigma H, \sigma \neq 0\}$, $f_{17} = \{H, D\}$, $f_{18} = \{D, C\}$, $f_{19} = \{J_3 + \eta D, H, \eta \neq 0\}$, $f_{20} = \{J_3 + \eta D, C, \eta \neq 0\}$
1	$f_{21} = \{J_3\}$, $f_{22} = \{H\}$, $f_{23} = \{D\}$, $f_{24} = \{C\}$, $f_{25} = \{C + \sigma H, \sigma \neq 0\}$, $f_{26} = \{J_3 + \tau H, \tau \neq 0\}$, $f_{27} = \{J_3 + \eta D, \eta \neq 0\}$, $f_{28} = \{J_3 + \rho C, \rho \neq 0\}$, $f_{29} = \{J_3 + \tau(C + \sigma H), \sigma, \tau \neq 0\}$

3.2. *Non-equivalent maximal subalgebras of* sch_3

With the preceding results, we can see that the only maximal subalgebras of $so(3)\oplus sl(2,\mathbb{R})$ are the algebras f_1, f_2, f_3, f_7 and f_8. Moreover, $n\equiv\{\boldsymbol{K},\boldsymbol{P}\}$ and $n_0=\{0\}$ are evidently two particular invariant subspaces for n so that the algebras $n_0 \square f_1$, $n\square f_2$, $n\square f_3$, $n\square f_7$ and $n\square f_8$ are splitting subalgebras of sch_3. These are the only non-equivalent maximal subalgebras of sch_3. Indeed all the other splitting or non-splitting ones are subalgebras of at least one among the maximal ones. These algebras, their corresponding dimensions and their bases of generators are listed in table 2.

Table 2. Maximal subalgebras of sch_3.

Notation	Dimension	Generators
f_1	6	$\{J, H, D, C\}$
$n\square f_2$	11	$\{J, H, D, \boldsymbol{K}, \boldsymbol{P}\}$
$n\square f_3$	11	$\{J, D, C, \boldsymbol{K}, \boldsymbol{P}\}$
$n\square f_7$	10	$\{J, C+\sigma H, \boldsymbol{K}, \boldsymbol{P}, \sigma\neq 0\}$
$n\square f_8$	10	$\{J_3, H, D, C, \boldsymbol{K}, \boldsymbol{P}\}$

Let us make a few comments about the structure of these algebras. The first one is the algebra $f=so(3)\oplus sl(2,\mathbb{R})$ itself. Such an algebra also appears as maximal into the conformal algebra. It has already received much attention in the literature in the Schrödinger (Jackiw 1980, Horvathy 1983, D'Hoker and Vinet 1984, 1985) as well as in the conformal (Beckers *et al* 1978) contexts. For the second one $n\square f_2$ we recognise the semi-direct sum of the Galilei algebra with the one-dimensional algebra $\{D\}$ associated with dilations. It appears maximal in the Schrödinger algebra as the similitude algebra sim(3,1) is maximal (Beckers *et al* 1978) in the algebra of the conformal group of spacetime. The third one, denoted $n\square f_3$, appears here as non-equivalent to the previous one since we have conjugated by the Galilei group and not by the Schrödinger group itself. The fourth subalgebra is clearly $n\square(so(3)\oplus so(2))$ while the fifth one is $n\square(so(2)\oplus sl(2,\mathbb{R}))$.

In correspondence with these five maximal subalgebras obtained in table 2, we get by exponentiation five maximal subgroups respectively denoted $G_1\equiv SO(3)\otimes SL(2,\mathbb{R})$, G_2, G_3, G_4, G_5 which will be of interest in § 4.

As a final comment, let us mention, besides the maximal subalgebras and subgroups, one of the six-dimensional non-maximal subalgebras of sch_3 generated by $\{J_3, \boldsymbol{K}, P_3, C\}$ which is contained in the two maximal subalgebras $n\square f_3$ and $n\square f_8$. The corresponding group will be denoted G_6 and will also be considered in the following sections.

4. Invariant tensor fields: global method

Let us now determine, by using the global method displayed in BHPW, invariant tensor fields under the five maximal subgroups of SCH_3 and one of its six-dimensional non-maximal subgroups (§§ 4.1–4.6). We intend to consider 1-forms, 2-forms and symmetric (0, 2)-tensor fields. Due to the linear action of SCH_3 on $\mathbb{R}^5$, we first determine such invariant fields on (at most four-dimensional) submanifolds of $\mathbb{R}^5$ and then project them back to the Newtonian spacetime $\mathbb{R}^{3+1}$.

Let us now very briefly review the global method (Beckers *et al* 1978) in order to introduce the notations in the Schrödinger context. Let p_0 be a generic point in $\mathbb{R}^5$ and G a subgroup of SCH_3. The orbit of G through p_0 is the submanifold of $\mathbb{R}^5$ defined by

$$Gp_0 = \{p \in \mathbb{R}^5 | \exists g \in G\colon p = gp_0\}. \tag{4.1}$$

Here, the orbit structure of $\mathbb{R}^{3+1}$ under the action of G is particularly simple: up to singular (lower-dimensional) submanifolds, the whole space is actually the orbit of G through p_0. In the following, we thus only consider the global method for tensor fields defined on one orbit. Let G_0 be the isotropy subgroup of G at p_0, i.e.

$$G_0 = \{g \in G | gp_0 = p_0\}. \tag{4.2}$$

With respect to the coordinate system $\{u^a\}$ introduced in § 2, all G-invariant (r, s)-tensor fields ψ on Gp_0 in $\mathbb{R}^5$ will be obtained by carrying out the following two steps.

(i) Solve the isotropy condition at p_0: $\forall g_0 \in G_0$

$$\psi^{i_1\ldots i_r}_{j_1\ldots j_s}(p_0) = (g_0)^{i_1}_{k_1}\ldots(g_0)^{i_r}_{k_r}(g_0^{-1})^{l_1}_{j_1}\ldots(g_0^{-1})^{l_s}_{j_s}\psi^{k_1\ldots k_r}_{l_1\ldots l_s}(p_0). \tag{4.3}$$

(ii) Apply the group transformation to get the field at any point of the orbit:

$$\psi^{i_1\ldots i_r}_{j_1\ldots j_s}(p) = \psi^{i_1\ldots i_r}_{j_1\ldots j_s}(gp_0) = (g)^{i_1}_{k_1}\ldots(g)^{i_r}_{k_r}(g^{-1})^{l_1}_{j_1}\ldots(g^{-1})^{l_s}_{j_s}\psi^{k_1\ldots k_r}_{l_1\ldots l_s}(p_0). \tag{4.4}$$

The tensor fields we are interested in can be respectively written as

$$A = A_a\,du^a \qquad F = F_{ab}\,du^a \wedge du^b \qquad S = S_{ab}\,du^a\,du^b \qquad (a, b = 1, \ldots, 5). \tag{4.5}$$

Using an obvious matrix notation for A, F and S, equations (4.3) and (4.4) become

$$A(p_0) = g_0^{-1T}A(p_0) \qquad F(p_0) = g_0^{-1T}F(p_0)g_0^{-1} \qquad S(p_0) = g_0^{-1T}S(p_0)g_0^{-1} \tag{4.6}$$

and

$$\begin{aligned} &A(p) = A(gp_0) = g^{-1T}A(p_0) \qquad F(p) = F(gp_0) = g^{-1T}F(p_0)g^{-1} \\ &S(p) = S(gp_0) = g^{-1T}S(p_0)g^{-1}. \end{aligned} \tag{4.7}$$

Let us consider the different cases corresponding to the subgroups $G_1, \ldots, G_6$ and let us only enter into the details of the case $G_1 \equiv SO(3) \otimes SL(2, \mathbb{R})$.

4.1. $G_1 = SO(3) \otimes SL(2, \mathbb{R})$

An element g of G_1 is given by (2.1) where $T = 0$. Choosing $p_0 = (1\ 0\ 0\ 0\ 1)^T$, an arbitrary point p of the orbit is given by

$$p = gp_0 = \begin{pmatrix} \hat{n} \\ \beta \\ \delta \end{pmatrix} \sim \begin{pmatrix} \hat{n}/\delta \\ \beta/\delta \\ 1 \end{pmatrix} \tag{4.8}$$

where the unit vector $\hat{n}$ is the first column of R. The isotropy subgroup G_0 contains all the elements $g_0 \in G_1$ such that

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}. \tag{4.9}$$

In $\mathbb{R}^5$, the orbit is defined by

$$(u^1)^2+(u^2)^2+(u^3)^2=1 \qquad u^5\neq 0. \tag{4.10}$$

The cotangent space for the orbit is then given by

$$u^1\,\mathrm{d}u^1+u^2\,\mathrm{d}u^2+u^3\,\mathrm{d}u^3=0 \tag{4.11}$$

which implies, in particular, that

$$\mathrm{d}u^1|p_0=0. \tag{4.12}$$

Using (2.3) and (4.10), we easily get the correspondence between the orbit in $\mathbb{R}^5$ and $\mathbb{R}^{3+1}$:

$$u^5=1/r \qquad u^4=t/r \qquad u^i=x^i/r \qquad (i=1,2,3) \tag{4.13}$$

where $r^2=\Sigma_{i=1}^3(x^i)^2$.

Let us now determine the invariant 1-forms. With (4.12), we have

$$A(p_0)=\sum_{i=2}^{5}A_i\,\mathrm{d}u^i. \tag{4.14}$$

Inserting (4.14) in (4.6) with g_0 given by (4.9) yields the solution at p_0:

$$A(p_0)=M\,\mathrm{d}u^4 \tag{4.15}$$

for some constant M. Introducing (4.15) in (4.7) yields

$$A(p)=M(\delta\,\mathrm{d}u^4-\beta\,\mathrm{d}u^5)=M(u^5\,\mathrm{d}u^4-u^4\,\mathrm{d}u^5) \tag{4.16}$$

where we have used (4.8). Now we project A on $\mathbb{R}^{3+1}$ by using (4.13) and finally obtain

$$A(x,t)=M\{(1/r)[(\mathrm{d}t/r)-(t/r^3)\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}]+(t/r^4)\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}\}=M\,\mathrm{d}t/r^2. \tag{4.17}$$

The invariant 2-forms F can also be easily determined. With (4.12), $F(p_0)$ does not contain any component along $\mathrm{d}u^1$ and (4.6) yields

$$F(p_0)=M_1\,\mathrm{d}u^2\wedge\mathrm{d}u^3+M_2\,\mathrm{d}u^4\wedge\mathrm{d}u^5 \tag{4.18}$$

for some constants M_1 and M_2. Choosing a parametrisation for the rotation matrix R, the evolution law (4.7) yields

$$F(p)=M_1\varepsilon_{ijk}u^i\,\mathrm{d}u^j\wedge\mathrm{d}u^k+M_2\,\mathrm{d}u^4\wedge\mathrm{d}u^5. \tag{4.19}$$

The projection on $\mathbb{R}^{3+1}$ gives the invariant 2-forms

$$F(x,t)=(M_1/r^3)\varepsilon_{ijk}x^i\,\mathrm{d}x^j\wedge\mathrm{d}x^k+(M_2/r^4)\,\mathrm{d}t\wedge(\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}). \tag{4.20}$$

Finally, starting with a symmetric matrix $S(p_0)$, (4.6) yields

$$S(p_0)=M_3[(\mathrm{d}u^2)^2+(\mathrm{d}u^3)^2]+M_4(\mathrm{d}u^4)^2 \tag{4.21}$$

for some constants M_3 and M_4, while (4.7) yields by using (4.11)

$$S(p)=M_3[(\mathrm{d}u^1)^2+(\mathrm{d}u^2)^2+(\mathrm{d}u^3)^2]+M_4(u^5\,\mathrm{d}u^4-u^4\,\mathrm{d}u^5)^2. \tag{4.22}$$

The projection on $\mathbb{R}^{3+1}$ gives the invariant symmetric $(0,2)$-tensor fields:

$$S(x,t)=M_3\{[(\mathrm{d}\boldsymbol{x})^2/r^2]-[(\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x})^2/r^4]\}+M_4\,\mathrm{d}t^2/r^4. \tag{4.23}$$

From now on, we only give the essential features of the other cases.

4.2. G_2

An element $g \in G_2$ is given by (2.1) where

$$L = \begin{pmatrix} \varepsilon & \beta \\ 0 & 1/\varepsilon \end{pmatrix}. \tag{4.24}$$

Choosing $p_0 = (0\ \ 0\ \ 0\ \ 0\ \ 1)^{\mathrm{T}}$, the orbit is given by

$$p = gp_0 = \begin{pmatrix} \boldsymbol{a} \\ \beta \\ 1/\varepsilon \end{pmatrix} \sim \begin{pmatrix} \varepsilon \boldsymbol{a} \\ \varepsilon\beta \\ 1 \end{pmatrix} \tag{4.25}$$

where $\boldsymbol{a}$ is the second column of T. The elements g_0 of the isotropy subgroup G_0 are such that $\boldsymbol{a} = \boldsymbol{0}$, $\beta = 0$ and the isotropy conditions (4.6) imply that all invariant tensor fields vanish.

4.3. G_3

An element $g \in G_3$ is given by (2.1) where

$$L = \begin{pmatrix} \varepsilon & 0 \\ \gamma & 1/\varepsilon \end{pmatrix}. \tag{4.26}$$

Choosing $p_0 = (0\ \ 0\ \ 0\ \ 1\ \ 1)^{\mathrm{T}}$, the orbit is given by

$$p = gp_0 = \begin{pmatrix} \boldsymbol{v} + \boldsymbol{a} \\ \varepsilon \\ \gamma + (1/\varepsilon) \end{pmatrix} \sim \begin{pmatrix} (\boldsymbol{v} + \boldsymbol{a})/[\gamma + (1/\varepsilon)] \\ \varepsilon/[\gamma + (1/\varepsilon)] \\ 1 \end{pmatrix} \tag{4.27}$$

where $\boldsymbol{v}$ and $\boldsymbol{a}$ are the columns of T. The elements g_0 of the isotropy subgroup G_0 are such that $\boldsymbol{a} = -\boldsymbol{v}$ and $\gamma = \varepsilon - (1/\varepsilon)$. Here too, the isotropy conditions (4.6) imply that all fields vanish.

4.4. G_4

An element $g \in G_4$ is now given by (2.1) where

$$L = \begin{pmatrix} \cos(\sqrt{\sigma}\varphi) & -\sqrt{\sigma}\sin(\sqrt{\sigma}\varphi) \\ (1/\sqrt{\sigma})\sin(\sqrt{\sigma}\varphi) & \cos(\sqrt{\sigma}\varphi) \end{pmatrix} \qquad \sigma \neq 0. \tag{4.28}$$

Choosing $p_0 = (0\ \ 0\ \ 0\ \ 0\ \ 1)^{\mathrm{T}}$, an arbitrary point of the orbit is given by

$$p = gp_0 = \begin{pmatrix} \boldsymbol{a} \\ -\sqrt{\sigma}\sin(\sqrt{\sigma}\varphi) \\ \cos(\sqrt{\sigma}\varphi) \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{a}/\cos(\sqrt{\sigma}\varphi) \\ -\sqrt{\sigma}\tan(\sqrt{\sigma}\varphi) \\ 1 \end{pmatrix}. \tag{4.29}$$

The isotropy subgroup G_0 contains all the elements $g_0 \in G_4$ such that $\boldsymbol{a} = \boldsymbol{0}$ and $L = \mathbb{1}$. In $\mathbb{R}^5$, the orbit is defined by

$$[(u^4)^2/\sigma] + (u^5)^2 = 1 \qquad u^5 \neq 0 \tag{4.30}$$

and the cotangent space for the orbit is given by

$$(u^4/\sigma)\,\mathrm{d}u^4 + u^5\,\mathrm{d}u^5 = 0 \tag{4.31}$$

which, in particular, implies

$$du^5|_{p_0}=0. \tag{4.32}$$

The correspondence between the orbit in $\mathbb{R}^5$ and $\mathbb{R}^{3+1}$ is

$$u^5=[1+(t^2/\sigma)]^{-1/2} \qquad u^4=t[1+(t^2/\sigma)]^{-1/2} \qquad u^i=x^i[1+(t^2/\sigma)]^{-1/2}. \tag{4.33}$$

The invariant tensor fields under study are then

$$A(x, t)=M\,dt/[1+(t^2/\sigma)] \tag{4.34}$$

$$F(x, t)=0 \tag{4.35}$$

$$S(x, t)=N\,dt^2/[1+(t^2/\sigma)]^2 \tag{4.36}$$

where M and N are arbitrary constants.

4.5. G_5

An element $g\in G_5$ is given by (2.1) where

$$R=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{4.37}$$

Choosing $p_0=(0\ \ 0\ \ 0\ \ 0\ \ 1)^T$, the orbit is given by

$$p=gp_0=\begin{pmatrix}\boldsymbol{a}\\ \beta\\ \delta\end{pmatrix}\sim\begin{pmatrix}\boldsymbol{a}/\delta\\ \beta/\delta\\ 1\end{pmatrix} \tag{4.38}$$

so that the elements g_0 of the isotropy subgroup G_0 are given by $\boldsymbol{a}=\boldsymbol{0}$, $\beta=0$ and $\delta=1/\varepsilon$. Once more, the isotropy conditions (4.6) imply that all invariant tensor fields vanish.

4.6. G_6

An element $g\in G_6$ is of the form (2.1) where

$$R=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix} \qquad L=\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix} \qquad \boldsymbol{a}=\begin{pmatrix}0\\ 0\\ \rho\end{pmatrix}. \tag{4.39}$$

Choosing p_0 as $(0\ \ 0\ \ 0\ \ 1\ \ 1)^T$, an arbitrary point p on the orbit is of the form:

$$p=gp_0=\begin{pmatrix}v_1\\ v_2\\ v_3+\rho\\ 1\\ \gamma+1\end{pmatrix}\sim\begin{pmatrix}v_1/(\gamma+1)\\ v_2/(\gamma+1)\\ (v_3+\rho)/(\gamma+1)\\ 1/(\gamma+1)\\ 1\end{pmatrix} \tag{4.40}$$

and the isotropy subgroup G_0 contains all the elements $g_0\in G_6$ such that $\boldsymbol{v}=(0, 0, -\rho)^T$ and $L=\mathbb{1}$. The orbit in $\mathbb{R}^5$ is defined by

$$u^4=1 \tag{4.41}$$

so that its cotangent space is given by

$$du^4 = 0 \tag{4.42}$$

and the correspondence between the orbit in $\mathbb{R}^5$ and $\mathbb{R}^{3+1}$ is

$$u^5 = 1/t \qquad u^i = x^i/t. \tag{4.43}$$

The invariant tensor fields are then

$$A(\boldsymbol{x}, t) = M \, dt/t^2 \tag{4.44}$$

$$F(\boldsymbol{x}, t) = M_1[(dx \wedge dy/t^2) + (dt/t^3) \wedge (y \, dx - x \, dy)] \tag{4.45}$$

$$S(\boldsymbol{x}, t) = (M_2/t^2)(dx^2 + dy^2) - (2M_2/t^3)(x \, dx + y \, dy) \, dt + (1/t^4)[M_2(x^2+y^2) + M_3] \, dt^2 \tag{4.46}$$

where M, M_1, M_2 and M_3 are arbitrary constants.

5. Invariant electromagnetic fields and potentials: infinitesimal method

In this section we introduce what we call 'Schrödinger's electromagnetism', i.e. Galilei's electromagnetism where besides Galilean transformations we include dilations and expansions in order to study the more general non-relativistic coordinate transformations leading to a group structure, as is clear from § 2. Then, within this Schrödinger electromagnetism, we want to establish invariant 2- and 1-forms through the infinitesimal method (analogous to that developed in BHPW) and to relate the results with the ones obtained in § 4. Let us first recall some characteristics of Galilei's electromagnetism (§ 5.1) and extend these considerations to Schrödinger's electromagnetism (§ 5.2).

5.1. Galilei's electromagnetism

The magnetic limit (Le Bellac and Lévy-Leblond 1973) of the Maxwell theory has already been studied. In such a limit, the electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{B}$) fields satisfy the following set of equations:

$$\nabla \times \boldsymbol{E} + \partial_t \boldsymbol{B} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{5.1}$$

and

$$\nabla \times \boldsymbol{B} = \boldsymbol{j} \qquad \nabla \cdot \boldsymbol{E} = \rho \tag{5.2}$$

where ρ and $\boldsymbol{j}$ are the charge and current densities respectively. From equations (5.1), the 'electromagnetic' field $F \equiv (\boldsymbol{E}, \boldsymbol{B})$ derives from scalar (V) and vector ($\boldsymbol{A}$) potentials as in the relativistic context:

$$\boldsymbol{E} = -\nabla V - \partial_t \boldsymbol{A} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{5.3}$$

and the associated transformation laws (under infinitesimal Galilean transformations) are (Le Bellac and Lévy-Leblond 1973):

$$\begin{aligned} \boldsymbol{E}'(\boldsymbol{x}', t') &= \boldsymbol{E}(\boldsymbol{x}, t) + \boldsymbol{\theta} \times \boldsymbol{E}(\boldsymbol{x}, t) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{x}, t) \\ \boldsymbol{B}'(\boldsymbol{x}', t') &= \boldsymbol{B}(\boldsymbol{x}, t) + \boldsymbol{\theta} \times \boldsymbol{B}(\boldsymbol{x}, t) \end{aligned} \tag{5.4}$$

while the corresponding ones on potentials (and currents) are

$$\begin{aligned} V'(\boldsymbol{x}', t') &= V(\boldsymbol{x}, t) - \boldsymbol{v} \cdot \boldsymbol{A}(\boldsymbol{x}, t) \\ \boldsymbol{A}'(\boldsymbol{x}', t') &= \boldsymbol{A}(\boldsymbol{x}, t) + \boldsymbol{\theta} \times \boldsymbol{A}(\boldsymbol{x}; t). \end{aligned} \tag{5.5}$$

If invariance conditions on *constant and uniform* electric and magnetic fields are under consideration, it is well known (Bacry *et al* 1970) that the kinematical group of such F is of dimension 6 and is the largest symmetry group of non-trivial invariant F. From the classification of all non-equivalent subalgebras of the Galilei algebra (Sorba 1974), it is easy to show that among the sixteen six-dimensional subalgebras only one of these leads to a non-trivial 'electromagnetic' field in the magnetic limit: it corresponds to the one leading to the above kinematical group and is explicitly given by the basis $\{J_3, K_3, \boldsymbol{P}, H\}$ leaving the so-called $F_{\|}$ invariant:

$$F_{\|} \equiv \{\boldsymbol{E} = (0, 0, E), \boldsymbol{B} \equiv (0, 0, B)\}. \tag{5.6}$$

Let us note that the above remark leads to a quite different result in comparison with the one of the Poincaré context (Combe and Sorba 1975) where there are non-trivial non-constant electromagnetic fields admitting a six-dimensional symmetry.

Finally, let us recall (Hussin 1984) in this Galilei electromagnetism that invariance conditions on *constant and uniform* '4-vectors' have also been considered. The corresponding kinematical groups are ten- or seven-dimensional structures depending on whether the spatial part is zero or not. These are the largest symmetries of arbitrary '4-vectors'.

5.2. *Schrödinger electromagnetism*

If we add to Galilean transformations the so-called dilations and expansions (see § 2), we need to extend the transformation laws (5.4) and (5.5). Under infinitesimal transformations (2.6), it is not difficult (Hussin and Sinzinkayo 1985) to obtain the new laws

$$\begin{aligned} \boldsymbol{E}'(\boldsymbol{x}', t') &= [1 - \tfrac{3}{2}(\alpha + 2ct)]\boldsymbol{E}(\boldsymbol{x}, t) + \boldsymbol{\theta} \times \boldsymbol{E}(\boldsymbol{x}, t) + (\boldsymbol{v} - c\boldsymbol{x}) \times \boldsymbol{B}(\boldsymbol{x}, t) \\ \boldsymbol{B}'(\boldsymbol{x}', t') &= [1 - (\alpha + 2ct)]\boldsymbol{B}(\boldsymbol{x}, t) + \boldsymbol{\theta} \times \boldsymbol{B}(\boldsymbol{x}, t) \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} V'(\boldsymbol{x}', t') &= [1 - (\alpha + 2ct)] V(\boldsymbol{x}, t) - (\boldsymbol{v} - c\boldsymbol{x}) \cdot \boldsymbol{A}(\boldsymbol{x}, t) \\ \boldsymbol{A}'(\boldsymbol{x}', t') &= [1 - \tfrac{1}{2}(\alpha + 2ct)]\boldsymbol{A}(\boldsymbol{x}, t) + \boldsymbol{\theta} \times \boldsymbol{A}(\boldsymbol{x}, t) \end{aligned} \tag{5.8}$$

ensuring the invariance of equations (5.1) and (5.2) under such Schrödinger transformations if the charge and current densities transform according to

$$\begin{aligned} \rho'(\boldsymbol{x}', t') &= [1 - 2(\alpha + 2ct)]\rho(\boldsymbol{x}, t) - (\boldsymbol{v} - c\boldsymbol{x}) \cdot \boldsymbol{j}(\boldsymbol{x}, t) \\ \boldsymbol{j}'(\boldsymbol{x}', t') &= [1 - \tfrac{3}{2}(\alpha + 2ct)]\boldsymbol{j}(\boldsymbol{x}, t) + \boldsymbol{\theta} \times \boldsymbol{j}(\boldsymbol{x}, t). \end{aligned} \tag{5.9}$$

We point out that under the Schrödinger group, the potential fields V and $\boldsymbol{A}$ transform like the derivatives (∂_t and $-\boldsymbol{\nabla}$) but not the densities ρ and $\boldsymbol{j}$. In the relativistic conformal context this result is completely similar due to the fact that $(V, \boldsymbol{A})$ and $(\rho, \boldsymbol{j})$ are not the same objects geometrically speaking: $(V, \boldsymbol{A})$ is a 1-form while $(\rho, \boldsymbol{j})$ appears as a 3-form in Minkowski spacetime.

Inside the Schrödinger context and more particularly inside this Schrödinger electromagnetism, we can get invariance conditions on 'electromagnetic' fields and potentials. These conditions are expressed in infinitesimal form:

$$\delta \boldsymbol{E}(\boldsymbol{x}, t) \equiv \tfrac{3}{2}(\alpha + 2ct)\boldsymbol{E} - \boldsymbol{\theta} \times \boldsymbol{E} - (\boldsymbol{v} - c\boldsymbol{x}) \times \boldsymbol{B} + \mathscr{D}\boldsymbol{E} = 0 \tag{5.10}$$

and

$$\delta \boldsymbol{B}(\boldsymbol{x}, t) \equiv (\alpha + 2ct)\boldsymbol{B} - \boldsymbol{\theta} \times \boldsymbol{B} + \mathscr{D}\boldsymbol{B} = 0 \tag{5.11}$$

where

$$\mathscr{D} = b\partial_t + \boldsymbol{a} \cdot \boldsymbol{\nabla} + \boldsymbol{\theta} \cdot (\boldsymbol{x} \times \boldsymbol{\nabla}) - t\boldsymbol{v} \cdot \boldsymbol{\nabla} + \alpha(t\partial_t + \tfrac{1}{2}\boldsymbol{x} \cdot \boldsymbol{\nabla}) + ct(t\partial_t + \boldsymbol{x} \cdot \boldsymbol{\nabla}). \tag{5.12}$$

For constant and uniform fields, we immediately obtain from (5.10):

$$(\alpha + 2ct)\boldsymbol{B}^2 = 0 \tag{5.13}$$

so that if we deal with the magnetic limit, we have to choose $\alpha = c = 0$. In conclusion, the symmetry is the one of Galilei's electromagnetism. The invariance conditions on potentials are

$$\delta V(\boldsymbol{x}, t) \equiv (\alpha + 2ct)V + (\boldsymbol{v} - c\boldsymbol{x}) \cdot \boldsymbol{A} + \mathscr{D}V = 0$$

and

$$\delta \boldsymbol{A}(\boldsymbol{x}, t) \equiv \tfrac{1}{2}(\alpha + 2ct)\boldsymbol{A} - \boldsymbol{\theta} \times \boldsymbol{A} + \mathscr{D}\boldsymbol{A} = 0. \tag{5.14}$$

Then in the constant and uniform case, we get the same results as in the Galilei context.

Let us now determine the fields (($\boldsymbol{E}$, $\boldsymbol{B}$) and (V, $\boldsymbol{A}$)) invariant under the maximal subalgebras (listed in table 2). We immediately deduce that only the maximal subalgebra $\mathrm{f}_1 \equiv \mathrm{so}(3) \oplus \mathrm{sl}(2, \mathbb{R})$ can admit a non-trivial 'electromagnetic' field. This case is very well known (Jackiw 1980, Horvathy 1983, D'Hoker and Vinet 1984, 1985): the most general field is

$$\boldsymbol{E} = E\boldsymbol{x}/r^4 \qquad \boldsymbol{B} = B\boldsymbol{x}/r^3. \tag{5.15}$$

Otherwise, only two of the maximal subalgebras admit non-trivial invariant '4-vectors'. Indeed, for the subalgebra f_1 we have

$$V = d/r^2 \qquad \boldsymbol{A} = 0 \tag{5.16}$$

while for the subalgebra $n \,\square\, \mathrm{f}_7$, we obtain

$$V = d'/(\sigma + t^2) \qquad \boldsymbol{A} = 0. \tag{5.17}$$

These results are the same as those obtained in § 4.

Finally, if we consider the six-dimensional non-maximal subalgebra of sch_3 generated by $\{J_3, C, \boldsymbol{K}, P_3\}$, it does admit an invariant 'electromagnetic' field which is explicitly obtained in the form

$$\boldsymbol{E} = (-my/t^3, mx/t^3, 0) \qquad \boldsymbol{B} = (0, 0, m/t^2). \tag{5.18}$$

Let us make some comments about the physical interpretation of the fields (5.15) and (5.18) (which are solutions of equations (5.1) and (5.2)). Firstly, let us insist on the fact that such 'electromagnetic' non-constant fields admit symmetry algebras of maximal dimension equal to six inside the Schrödinger algebra. Secondly, for the field (5.15), we evidently recognise the field of the magnetic monopole—already discussed in the Schrödinger context (Jackiw 1980, Horvathy 1983, D'Hoker and Vinet 1984, 1985)—and an electric field deriving from an r^{-2} scalar potential (D'Hoker and Vinet 1984, 1985). Finally, the invariant field (5.18) derives from the potential

$$V = 0 \qquad \boldsymbol{A} = (-my/2t^2, mx/2t^2, 0) \tag{5.19}$$

which admits a symmetry subalgebra of the field generated by $\{J_3, K_3, C, P_3\}$. In such a case, let us recall that for the missing field symmetries we can compensate the non-invariance by introducing so-called (Janner and Janssen 1971) compensating gauge transformations W defined by (Beckers and Hussin 1983a, b)

$$\partial_t W = -\delta V \qquad \nabla W = -\delta \boldsymbol{A}. \tag{5.20}$$

Here, the non-constant transformations W are explicitly

$$W_{K_1} = tA_1 \qquad W_{K_2} = tA_2. \tag{5.21}$$

6. Comments

We first notice that the results of §§ 4 and 5 are evidently in complete agreement. They show that no non-trivial F exists when the dimension of the symmetry group is greater than six, a property very easily deduced from the infinitesimal approach. In the G_1 and G_6 cases, we get two physically interesting results.

(i) The $G_1 = SO(3) \times SL(2, \mathbb{R})$ maximal subgroup gives the expected results at the level of the magnetic field (Jackiw 1980, Horvathy 1983, Hussin and Sinzinkayo 1985) and its magnetic monopole context as well as at the level of the electric field (D'Hoker and Vinet 1984, 1985, Hussin and Sinzinkayo 1985).

(ii) The G_6 subgroup, a non-maximal one but a common subgroup (of dimension 6) of two maximal ones (G_3 and G_5), is a specific case showing that there exist invariant *non-constant* electromagnetic fields other than Coulomb-like fields which satisfy the equations of Schrödinger's electromagnetism.

Secondly, we insist on the G_6 case from the point of view of compensating gauge transformations, subsymmetries of the potentials with respect to those of the field and, consequently, on non-trivial extensions in correspondence with the explicit forms of constants of motion (Beckers and Hussin 1984). In such a context, let us mention the six constants of motion issued from the corresponding realisation of the extended subalgebra. These elements are based on the results (5.19)-(5.21) and we obtain

$$\begin{aligned} &J_3 = (\boldsymbol{x} \times \boldsymbol{p})_3 + \tfrac{1}{2}\mathrm{i}\sigma_3 && K_3 = -tp_3 + mz \\ &K_1 = -tp_1 + mx + eW_{K_1} && K_2 = -tp_2 + my + eW_{K_2} \\ &P_3 = p_3 && C = t[tH_{\mathrm{P}} - t\boldsymbol{x} \cdot \boldsymbol{p} + (3\mathrm{i}/2)] + (m/2)\boldsymbol{x}^2 \end{aligned} \tag{6.1}$$

where H_{P} is the Pauli Hamiltonian

$$H_{\mathrm{P}} = (1/2m)(\boldsymbol{p} + e\boldsymbol{A})^2 - (e/2m)\boldsymbol{B} \cdot \boldsymbol{\sigma}. \tag{6.2}$$

Thirdly, let us just mention that if invariant (0, 2)-symmetric tensors S can be determined from the global point of view (see § 4), we know that in the Newtonian spacetime they have no direct meaning as metric tensors in contradiction with respect to the relativistic cases developed in BHPW.

Finally, let us complete our comments in connection with Schrödinger's electromagnetism by pointing out that the so-called electric limit (Le Bellac and Lévy-Leblond 1973) can in principle be studied through the infinitesimal method as we have worked out the magnetic one, but not through the global method as is clear from geometrical considerations. Such a remark asks for complementary comments actually under study: let us only mention here that forms correspond to covariant tensors and are *ad hoc* geometrical objects with respect to the global approach when the magnetic limit is taken into consideration but not the electric limit.

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